ELLIPSES.

The third Bicircloid Curve, produced by two circular movements (in the same plane) when the angular velocities are as $2:1$, in contrary directions, has for its central equations

$$x = \cos\phi + e\cos\phi = (1+e)\cos\phi;$$
$$y = \sin\phi - e\sin\phi = (1-e)\sin\phi;$$

which, by putting $a = 1+e$ the apocentral radius, and $p = 1-e$ the pericentral radius, become

$$x = a\cos\phi, \text{ and } y = p\sin\phi.$$

Hence $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{p}\right)^2 = 1.$ . . . . (1)

Therefore the Curve is an Ellipse, of which a and p are the apsidal major and minor semi-axes, respectively.

The excentricity of this Ellipse is

$$a\epsilon = \sqrt{a^2 - p^2} = \sqrt{(a+p)(a-p)} =$$
$$\sqrt{(1+e+1-e)(1+e-1+e)} = \sqrt{2\times 2e} = 2\sqrt{e}. \quad (2)$$

Or, the excentricity = the distance from the center to either focus, is a mean proportional between the diameters of the Deferent and Epicycle.

Substituting $r\cos\theta$ for x, $r\sin\theta$ for y in (1.), and clearing from fractions, there results

$$p^2 r^2 \cos^2\theta + a^2 r^2 \sin^2\theta = a^2 p^2;$$

whence $r^2 = \dfrac{a^2 p^2}{p^2\cos^2\theta + a^2\sin^2\theta}$. . . . . (3)

Putting in the denominator, $a^2 - a^2\epsilon^2$ for p^2, there results

$$r^2 = \frac{a^2 p^2}{a^2 - a^2\epsilon^2\cos^2\theta}. \quad \ldots\ldots \quad (4)$$

Again, putting $\frac{1}{2}(1+\cos 2\theta)$ for $\cos^2\theta$, in the denominator of (4), there results.

$$\left.\begin{aligned} r^2 &= \frac{a^2 p^2}{\left(a^2 - \frac{1}{2}a^2\epsilon^2\right) - \frac{1}{2}a^2\epsilon^2\cos 2\theta} \\ &= \frac{2a^2p^2}{a^2 + p^2 - a^2\epsilon^2\cos 2\theta}. \end{aligned}\right\} \quad \ldots \quad (5)$$

In the equations (1), (3), (4), and (5), the center is the origin or pole.

The polar equation when the focus is the pole is

$$r=\frac{p^2}{a+a\,\epsilon\cos\theta}. \qquad (6)$$

When the Epicyclic radius vanishes, or $e=0$, then $a=p$; in this case equation (1) becomes

$$x^2+y^2=a^2=p^2; \qquad (7)$$

or

$$r=a=p. \qquad (8)$$

Hence in this case the curve is a circle, whose radius is a or p.

Similarly, equation (3) gives, as in (8),

$$r^2=\frac{a^4}{a^2\cos^2\theta+a^2\sin^2\theta}=a^2; \text{ or } r=a=p.$$

When the Epicyclic radius becomes equal to the Deferent radius, or $e=1$, then $a=2$, $p=0$; and equation (1) becomes

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{0}\right)^2=1; \quad \text{or } y=0.$$

The curve in this case becomes a straight line, coinciding with the axis of x.

Equation (3) gives $r=0$, since $p=0$; but here must be excepted the case of $\sin\theta=0$, in which case equation (3) becomes $r^2=\frac{a^2p^2}{p^2}=a^2$; $\therefore r=\pm a$; and these are the limiting values of r.

The polar equation of the third Bicircloid, when centric, gives $r=a\cos\frac{2\theta}{0}$; which, when $\theta=0$, or $\theta=180°$, gives, as before, $r=+a$, or $r=-a$, the limiting values.

This curve is the *Rectoid* or *Orthoid*; being a finite straight line, whose length is equal to the Apocentral Diameter.

The values $e=0$, and $e=1$, give the two extreme cases of the curve.

If we suppose e infinite, then $a=e$, $p=-e$; and we get, both at once, the two cases in one; for the curve becomes a circle: while, its radius being infinite, the circle is equivalent to a straight line: but here the length is not finite, because it is not centric. This, however, is an Imaginary Bicircloid.

Let $D=1$, $e=\frac{m}{n}$; then

$$p=\frac{n-m}{n};\quad a=\frac{n+m}{n}. \qquad (9)$$

$$a\epsilon=2\sqrt{e}, \text{ by (2) } \therefore a\epsilon=\frac{2\sqrt{mn}}{n};\ \epsilon=\frac{2\sqrt{mn}}{n+m}. \qquad (10)$$

Let $a=1$, $p=\frac{m}{n}$, then

$$D=\frac{n+m}{2n};\ E=\frac{n-m}{2n};\ a\epsilon=\epsilon=\frac{\sqrt{n^2-m^2}}{n}. \qquad (11)$$

Let $a=1$, $\epsilon=\frac{m}{n}$, then $p=\frac{\sqrt{n^2-m^2}}{n}$. $\qquad$ (12)

$$D=\frac{n+\sqrt{n^2-m^2}}{2n};\ E=\frac{n-\sqrt{n^2-m^2}}{2n}. \qquad (13)$$

The letter v, c, or f, attached as an index to the letters x and r, denotes that the origin, or the pole, is at the *vertex*, *center*, or *focus*, respectively.

Thus, when the center is the origin, we have $a^2y^2+p^2x_c^2=a^2p^2$; from equation (1).

Whence $a^2y^2+p^2=a^2-x_c^2$; which is the second form of the rectangular equation.

If we make the vertex the origin, put $x_v+x_c=a$; then $x_c=a-x_v$; and substituting this in equation (1), we have $a^2y^2+p^2=a^2-(a-x_v)^2=2ax_v-x_v^2$; as in the first form.

When the origin is at the focus, $x_f+x_c=a\epsilon$; $\therefore x_c=a\epsilon-x_f$; and

$$a^2y^2+p^2=a^2-(a\epsilon-x_f)^2=a^2-a^2\epsilon^2+2a\epsilon x_f-x_f^2$$
$$=p^2+2\sqrt{a^2-p^2}.x_f-x_f^2;$$

which is the third form.

The polar equations are deduced from the rectangular equations by making

$$x_v=r_v\cos\theta_v,\ y=r_v\sin\theta_v;$$

$$x_c=r_c\cos\theta_c,\ y=r_c\sin\theta_c;$$

$$x_f=r_f\cos\theta_f,\ y=r_f\cos\theta_f;$$

according as the pole is at the vertex, at the center, or at the focus, respectively.

Thus, we have

$$a^2 y^2 + p^2 = 2\,a\,x_v - x_v^2;$$

whence $\quad a^2 r_v^2 \sin^2\theta_v + p^2 = 2\,a\,r_v \cos\theta_v - r_v^2 \cos^2\theta_v,$

or $\quad a^2 r_v \sin^2\theta_v = 2\,a\,p^2 \cos\theta_v - p^2 r_v \cos^2\theta_v;$

$$\therefore r_v = \frac{2\,a\,p^2 \cos\theta_v}{a^2 \sin^2\theta_v + p^2 \cos^2\theta_v} = \frac{2\,a\,p^2 \cos\theta_v}{a^2 - (a^2 - p^2)\cos^2\theta_v};$$

and putting $\dfrac{1+\cos 2\,\theta_v}{2}$ for $\cos^2\theta_v$, we have

$$r_v = \frac{2\,a\,p^2 \cos\theta_v}{a^2 - \dfrac{a^2-p^2}{2} - \dfrac{a^2-p^2}{2}\cos 2\,\theta_v} =$$

$$\frac{4\,a\,p^2 \cos\theta_v}{2\,a^2 - a^2 + p^2 - (a^2 - p^2)\cos 2\,\theta_v} = \frac{2 \times 2\,a\,p^2 \cos\theta_v}{a^2 + p^2 - (a^2 - p^2)\cos 2\,\theta_v}.$$

If we put $\epsilon \cos\theta = \cos\psi$, then

$$a^2 \epsilon^2 \cos^2\theta = (a^2 - p^2)\cos^2\theta = a^2 \cos^2\psi;$$

and we have also

$$r_v = \frac{2\,a\,p^2 \cos\theta_v}{a^2 - a^2\cos^2\psi_v} = \frac{2\,a\,p^2 \cos\theta_v}{a^2 \sin^2\psi_v} = \frac{2\,p^2 \cos\theta_v}{a \sin^2\psi_v}.$$

Since $a^2 \epsilon^2 = a^2 - p^2$, equation (4) may be written

$$r_c^2 = \frac{a^2 p^2}{a^2 - (a^2 - p^2)\cos^2\theta_c};$$

and putting $a^2 \cos^2\psi$ for $(a^2 - p^2)\cos^2\theta_c$, we have

$$r_c^2 = \frac{a^2 p^2}{a^2 - a^2 \cos^2\psi_c} = \frac{a^4 p^2}{a^2 \sin^2\psi_c} = \frac{p^2}{\sin^2\psi_c}; \quad \therefore r_c = \frac{p}{\sin\psi_c}.$$

Similarly, equation (6) may be written

$$r_f = \frac{p^2}{a + \sqrt{a^2 - p^2}\cos\theta_f} = \frac{p^2}{a + a\cos\psi_f} = \frac{p^2}{a(1 + \cos\psi_f)} = \frac{p^2}{2\,a\cos^2\dfrac{\psi_f}{2}}.$$

The angle ψ may be geometrically constructed as follows: On the major axis of the ellipse, as a diameter, describe a semicircle. Project orthogonally on the radius vector, or its prolongation, the distance between the foci, *viz.* $2\,a\,\epsilon$. At the extremity of the major axis inscribe a chord in the semicircle, which shall be equal to the preceding projection of $2\,a\,\epsilon$; this chord makes with the major axis an angle $= \psi$.

If ϖ denote the perpendicular on the tangent, then according as the pole is taken at the *vertex*, *focus*, or *center*, so we have

$$\varpi_v = \frac{p}{a\,\varepsilon}\ \frac{\sqrt{p^4 + a^2\varepsilon^2 r_v^2} - p^2}{\sqrt{2\sqrt{p^4 + a^2\varepsilon^2 r_v^2} - p^2(2-\varepsilon^2) - \varepsilon^2 r_v^2}}.$$

$$\varpi_f = p\sqrt{\frac{r_f}{2a - r_f}}.$$

$$\varpi_c = \frac{a\,p}{\sqrt{a^2 + p^2 - r^2}}.$$

If x' and y' are the coordinates of a point in the curve which corresponds to the angle $\frac{\pi}{2} - \phi$; x'', y'', those corresponding to the angle $\phi - \frac{\pi}{2}$; and x''', y'', those corresponding to the angle $\phi + \frac{\pi}{2}$; then we have

$$x\,y' + x'\,y = x''\,y - x\,y'' = x\,y''' - x'''\,y = a\,p.$$

$$r^2 + r'^2 = r^2 + r''^2 = r^2 + r'''^2 = a^2 + p^2.$$

The relation between θ_c and ϕ is

$$(1 - \varepsilon^2\cos^2\theta_c) \times \left(1 + \frac{\varepsilon^2}{1-\varepsilon^2}\cos^2\phi\right) = 1.$$

Whence

$$\varepsilon\cos\phi\cos\theta_c = \sqrt{(\cos\phi + \sqrt{1-\varepsilon^2}\cos\theta_c)(\cos\phi - \sqrt{1-\varepsilon^2}\cos\theta_c)};$$

$$\cos\theta_c = \frac{\cos\phi}{\sqrt{1-\varepsilon^2\sin^2\phi}}, \quad \text{and} \quad \cos\phi = \frac{\sqrt{1-\varepsilon^2}\cos\theta_c}{\sqrt{1-\varepsilon^2\cos^2\theta_c}}.$$

Let u and w be the coordinates of the Evolute, respectively parallel to the coordinates x and y, then

$$u = \frac{4\,e}{1+e}\cos^3\phi = \frac{a^2\varepsilon^2}{a}\cos^3\phi, \quad w = -\frac{4\,e}{1-e}\sin^3\phi = -\frac{a^2\varepsilon^2}{p}\sin^3\phi,$$

whence $\quad (pw)^{\frac{2}{3}} + (a\,u)^{\frac{2}{3}} = (a\,\varepsilon)^{\frac{4}{3}}.$

and $\quad \sqrt[3]{pw} = \sqrt{(\sqrt[3]{a^2\varepsilon^2} + \sqrt[3]{a\,u})(\sqrt[3]{a^2\varepsilon^2} - \sqrt[3]{a\,u})}.$

The following are examples of **Ellipses** whose principal axes are proportioned, to each other, according to the series of ratios expressed by every pair of numbers not exceeding 10.

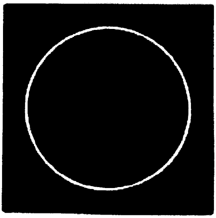

Fig. 1. Axes as 1 : 1.

Radii as 1 : 0; $e=0$.

$a=1, p=1$; $1^2\epsilon^2=1^2-1^2=0$.

The *foci* at the *center*.

A Circle, one extreme of the Ellipse: the other being the Orthoid, fig. 33.

$y^2=2x_v-x_v^2=1-x_c^2=1-x_f^2$. $r_v=2\cos\theta_v$. $r_c=r_f=1$.

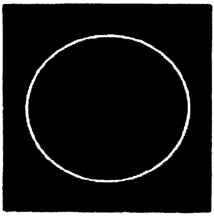

Fig. 2. Axes as 10 : 9.

Radii as 19 : 1; $e=0{\cdot}05263$.

$a=1, p=0{\cdot}9$; $10^2\epsilon^2=10^2-9^2=19$.

$$10^2y^2\div9^2=2.10\,x_v-x_v^2=10^2-x_c^2$$
$$=9^2+2\sqrt{10^2-9^2}.x_f-x_f^2.$$

$r_v=2.10.9^2\cos\theta_v\div\{10^2-(10^2-9^2)\cos^2\theta_v\}$

$=2.9^2\cos\theta_v\div10\sin^2\psi_v$. $r_c^2=10^2.9^2\div\{10^2-(10^2-9^2)\cos^2\theta_c\}$,

$r_c=9\div\sin\psi_c$. $r_f=9^2\div\{10+\sqrt{10^2-9^2}.\cos\theta_f\}=9^2\div2.10\cos^2\dfrac{\psi}{2}$

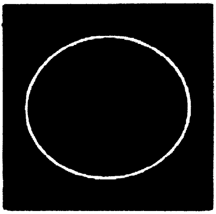

Fig. 3. Axes as 9 : 8.

Radii as 17 : 1; $e=0{\cdot}0588$.

$a=1$; $p=0{\cdot}8$; $9^2\epsilon^2=9^2-8^2=17$.

$$9^2y^2\div8^2=2.9x_v-x_v^2=9^2-x_c^2$$
$$=8^2+2\sqrt{9^2-8^2}.x_f-x_f^2.$$

$r_v=2.9.8^2\cos\theta_v\div\{9^2-(9^2-8^2)\cos^2\theta_v\}$

$=2.8^2\cos\theta_v\div9\sin^2\psi_v$. $r_c^2=9^2.8^2\div\{9^2-(9^2-8^2)\cos^2\theta_c\}$,

$r_c=8\div\sin\psi_c$. $r_f=8^2\div\{9+\sqrt{9^2-8^2}.\cos\theta_f\}=8^2\div2.9\cos^2\dfrac{\psi_f}{2}$.

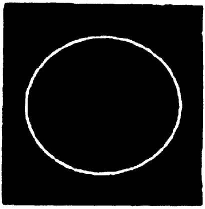

Fig. 4. Axes as 8 : 7.

Radii as 15 : 1 ; $e=0{\cdot}0\dot{6}$.

$a=1,\ p=0{\cdot}875$; $8^2\epsilon^2=8^2-7^2=15$.

$$8^2 y^2 \div 7^2 = 2.8\,x_v - x_v^2 = 8^2 - x_c^2$$
$$= 7^2 + 2\sqrt{8^2-7^2}.\,x_f - x_f^2.$$

$$r_v = 2.8.7^2 \cos\theta_v \div \{8^2-(8^2-7^2)\cos^2\theta_v\}$$
$$= 2.7^2\cos\theta_v \div 8\sin^2\psi_v. \quad r_c^2 = 8^2.7^2 \div \{8^2-(8^2-7^2)\cos^2\theta_c\},$$
$$r_c = 7 \div \sin\psi_c. \quad r_f = 7^2 \div \{8 + \sqrt{8^2-7^2}.\cos\theta_f\} = 7^2 \div 2.8\cos^2\frac{\psi_f}{2}.$$

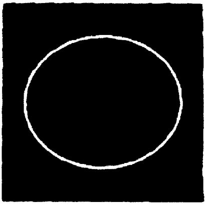

Fig. 5. Axes as 7 : 6.

Radii as 13 : 1 ; $e=0{\cdot}0769$.

$a=1,\ p=0{\cdot}\dot{8}5714\dot{2}$; $7^2\epsilon^2=7^2-6^2=13$.

$$7^2 y^2 \div 6^2 = 2.7\,x_v - x_v^2 = 7^2 - x_c^2$$
$$= 6^2 + 2\sqrt{7^2-6^2}.\,x_f - x_f^2.$$

$$r_v = 2.7.6^2 \cos\theta_v \div \{7^2-(7^2-6^2)\cos^2\theta_v\}$$
$$= 2.6^2\cos\theta_v \div 7\sin^2\psi_v. \quad r_c^2 = 7^2.6^2 \div \{7^2-(7^2-6^2)\cos^2\theta_c\},$$
$$r_c = 6 \div \sin\psi_c. \quad r_f = 6^2 \div \{7 + \sqrt{7^2-6^2}.\cos\theta_f\} = 6^2 \div 2.7\cos^2\frac{\psi_f}{2}.$$

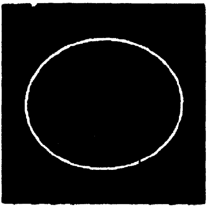

Fig. 6. Axes as 6 : 5.

Radii as 11 : 1; $e=0{\cdot}\dot{0}\dot{9}$.

$a=1,\ p=0{\cdot}8\dot{3}$; $6^2\epsilon^2=6^2-5^2=11$.

$$6^2 y^2 \div 5^2 = 2.6\,x_v - x_v^2 = 6^2 - x_c^2$$
$$= 5^2 + 2\sqrt{6^2-5^2}.\,x_f - x_f^2.$$

$$r_v = 2.6.5^2 \cos\theta_v \div \{6^2-(6^2-5^2)\cos^2\theta_v\}$$
$$= 2.5^2\cos\theta_v \div 6\sin^2\psi_v. \quad r_c^2 = 6^2.5^2 \div \{6^2-(6^2-5^2)\cos^2\theta_c\},$$
$$r_c = 5 \div \sin\psi_c. \quad r_f = 5^2 \div \{6 + \sqrt{6^2-5^2}.\cos\theta_f\} = 5^2 \div 2.6\cos^2\frac{\psi_f}{2}.$$

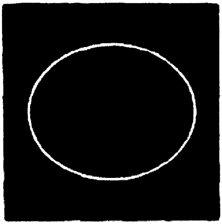

Fig. 7. Axes as 5 : 4.

Radii as 9 : 1 ; $e=0\dot{1}$.

$a=1, p=0{\cdot}8$; $5^2 e^2=5^2-4^2=9$.

$5^2 y^2 \div 4^2 = 2.5\, x_v - x_v^2 = 5^2 - x_c^2$

$= 4^2 + 2\sqrt{5^2-4^2}.x_f - x_f^2$.

$r_v = 2.5.4^2 \cos\theta_v \div \{5^2-(5^2-4^2)\cos^2\theta_v\}$

$= 2.4^2 \cos\theta_v \div 5 \sin\psi_v$. $r_c^2 = 5^2.4^2 \div \{5^2-(5^2-4^2)\cos^2\theta_c\}$,

$r_c = 4 \div \sin\psi_c$. $r_f = 4^2 \div \{5(+5^2-4^2)\cos\theta_f\} = 4^2 \div 2.5\cos^2\frac{\psi_f}{2}$

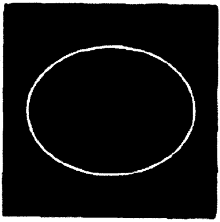

Fig. 8. Axes as 9 : 7.

Radii as 8 : 1 ; $e=0{\cdot}125$.

$a=1, p=0{\cdot}\dot{7}$; $9^2 e^2=9^2-7^2=32$.

$9^2 y^2 \div 7^2 = 2.9\, x_v - x_v^2 = 9^2 - x_c^2$

$= 7^2 + 2\sqrt{9^2-7^2}\, x_f - x_f^2$,

$r_v = 2.9.7^2 \cos\theta_v \div \{9^2-(9^2-7^2)\cos^2\theta_v\}$

$= 2.7^2 \cos\theta_v \div 9 \sin^2\psi_v$. $r_c^2 = 9^2.7^2 \div \{9^2-(9^2-7^2)\cos^2\theta_c\}$,

$r_c = 7 \div \sin\psi_c$. $r_f = 7^2 \div \{9 + \sqrt{9^2-7^2}.\cos\theta_f\} = 7^2 \div 2.9\cos^2\frac{\psi_f}{2}$

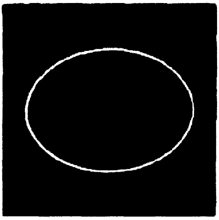

Fig. 9. Axes as 4 : 3.

Radii as 7 : 1 ; $e=0{\cdot}\dot{1}4285\dot{7}$.

$a=1, p=0{\cdot}75$; $4^2 e^2=4^2-3^2=7$.

$4^2 y^2 \div 3^2 = 2.4\, x_v - x_v^2 = 4^2 - x_c^2$

$= 3^2 + 2\sqrt{4^2-3^2}.x_f - x_f^2$.

$r_v = 2.4.3^2 \cos\theta_v \div \{4^2-(4^2-3^2)\cos^2\theta_v\}$

$= 2.3^2 \cos\theta_v \div 4 \sin^2\psi_v$. $r_c^2 = 4^2.3^2 \div \{4^2-(4^2-3^2)\cos^2\theta_c\}$,

$r_c = 3 \div \sin\psi$. $r_f = 3^2 \div \{4 + \sqrt{4^2-3^2}.\cos\theta_f\} = 3^2 \div 2.4\cos^2\frac{\psi_f}{2}$.

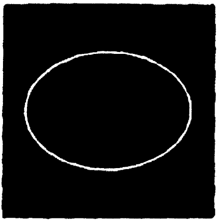

Fig. 10. Axes as 7 : 5.

Radii as 6 : 1; $e=0{\cdot}1\dot{6}$.

$a=1,\ p=0{\cdot}\dot{7}1428\dot{5};\ 7^2\epsilon^2=7^2-5^2=24.$

$7^2y^2\div5^2=2.7x_v-x_v^2=7^2-x_e^2$

$=5^2+2\sqrt{7^2-5^2}.x_f-x_f^2.$

$r_v=2.7.5^2\cos\theta_v\div\{7^2-(7^2-5^2)\cos^2\theta_v\}$

$=2.5^2\cos\theta_v\div7\sin^2\psi_v.\ \ r_c^2=7^2.5^2\div\{7^2-(7^2-5^2)\cos^2\theta_e\},$

$r_e=5\div\sin\psi_e.\ \ r_f=5^2\div\{7+\sqrt{7^2-5^2}.\cos\theta_f\}=5^2\div2.7\cos^2\frac{\psi_f}{2}.$

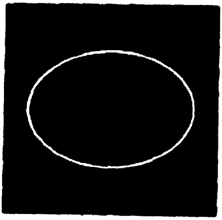

Fig. 11. Axes as 10 : 7.

Radii as 17 : 3; $e=0{\cdot}17647$.

$a=1,\ p=0{\cdot}7;\ 10^2\epsilon^2=10^2-7^2=51.$

$10^2y^2\div7^2=2.10x_v-x_v^2=10^2-x_e^2$

$=7^2+2\sqrt{10^2-7^2}.x_f-x_f^2.$

$r_v=2.10.7^2\cos\theta_v\div\{10^2-(10^2-7^2)\cos^2\theta_v\}$

$=2.7^2\cos\theta_v\div10\sin^2\psi_v.\ \ r_c^2=10^2.7^2\div\{10^2-(10^2-7^2)\cos^2\theta_e\},$

$r_e=7\div\sin\psi_e.\ \ r_f=7^2\div\{10+\sqrt{10^2-7^2}.\cos\theta_f\}=7^2-2.10\cos^2\frac{\psi_f}{2}.$

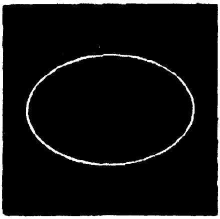

Fig. 12. Axes as 3 : 2.

Radii as 5 : 1; $e=0{\cdot}2$.

$a=1,\ p=0{\cdot}\dot{6};\ 3^2\epsilon^2=3^2-2^2=5.$

$3^2y^2\div2^2=2.3x_v-x_v^2=3^2-x_e^2$

$=2^2+2\sqrt{3^2-2^2}.x_f-x_f^2.$

$r_v=2.3.2^2\cos\theta_v\div\{3^2-(3^2-2^2)\cos^2\theta_v\}$

$=2.2^2\cos\theta_v\div3\sin^2\psi_v.\ \ r_c^2=3^2.2^2\div\{3^2-(3^2-2^2)\cos^2\theta_c\},$

$r_e=2\div\sin\psi_e.\ \ r_f=2^2\div\{3+\sqrt{3^2-2^2}.\cos\theta_f\}=2^2\div2.3\cos^2\frac{\psi_f}{2}.$

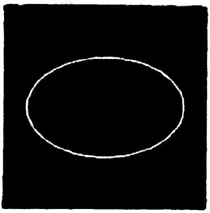

Fig. 13. Axes as 8 : 5.

Radii as 13 : 3 ; $e=0\cdot 23$.

$a=1, p=0\cdot625; 8^2\epsilon^2=8^2-5^2=39.$

$8^2y^2\div5^2=2.8x_v-x_v^2=8^2-x_c^2$

$=5^2+2\sqrt{8^2-5^2}.x_f-x_f^2.$

$r_v=2.8.5^2\cos\theta_v\div\{8^2-(8^2-5^2)\cos^2\theta_v\}$

$=2.5^2\cos\theta_v\div8\sin^2\psi_v.\quad r_c^2=8^2.5^2\div\{8^2-(8^2-5^2)\cos^2\theta_c\},$

$r_c=5\div\sin\psi_c.\quad r_f=5^2\div\{8+\sqrt{8^2-5^2}.\cos\theta_f\}=5^2\div2.8\cos^2\frac{\psi_f}{2}.$

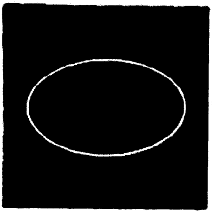

Fig. 14. Axes as 5 : 3.

Radii as 4 : 1 ; $e=0\cdot25$.

$a=1, p=0\cdot6; 5^2\epsilon^2=5^2-3^2=16.$

$5^2y^2\div3^2=2.5x_v-x_v^2=5^2-x_c^2$

$=3^2+2\sqrt{5^2-3^2}.x_f-x_f^2.$

$r_v=2.5.3^2\cos\theta_v\div\{5^2-(5^2-3^2)\cos^2\theta_v\}$

$=2.3^2\cos\theta_v\div5\sin^2\psi_v.\quad r_c^2=5^2.3^2\div\{5^2-(5^2-3^2)\cos^2\theta_c\},$

$r_c=3\div\sin\psi_c.\quad r_f=3^2\div\{5+\sqrt{5^2-3^2}.\cos\theta_f\}=3^2\div2.5\cos^2\frac{\psi_f}{2}.$

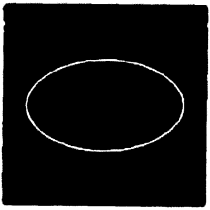

Fig. 15. Axes as 7 : 4.

Radii as 11 : 3 ; $e=0\cdot\dot{2}\dot{7}$.

$a=1, p=0\cdot\dot{5}7142\dot{8}; 7^2\epsilon^2=7^2-4^2=33.$

$7^2y^2\div4^2=2.7x_v-x_v^2=7^2-x_c^2$

$=4^2+2\sqrt{7^2-4^2}.x_f-x_f^2.$

$r_v=2.7.4^2\cos\theta_v\div\{7^2-(7^2-4^2)\cos^2\theta_v\}$

$=2.4^2\cos\theta_v\div7\sin^2\psi_v.\quad r_c^2=7^2.4^2\div\{7^2-(7^2-4^2)\cos^2\theta_c\},$

$r_c=4\div\sin\psi_c.\quad r_f=4^2\div\{7+\sqrt{7^2-4^2}.\cos\theta_f\}=4^2\div2.7\cos^2\frac{\psi_f}{2}.$

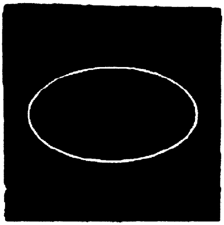

Fig. 16. Axes as 9 : 5.

Radii as 7 : 2 ; $e=0\cdot\dot{2}8571\dot{4}$.

$a=1,\ p=0\cdot\dot{5}$; $9^2\epsilon^2=9^2-5^2=56$.

$$9^2y^2\div5^2=2.9\,x_v-x_v^2=9^2-x_c^2$$

$$=5^2+2\sqrt{9^2-5^2}.\,x_f-x_f^2.$$

$$r_v=2.9.5^2\cos\theta_v\div.\,9^2-(9^2-5^2)\cos^2\theta_v\}$$

$$=2.5^2\cos\theta_v\div9\sin^2\psi_v.\quad r_c^2=9^2.5^2\div\{9^2-(9^2-5^2)\cos^2\theta_c\},$$

$$r_c=5\div\sin\psi_c.\quad r_f=5^2\div\{9+\sqrt{9^2-5^2}.\cos\theta_f\}=5^2\div2.9\cos^2\frac{\psi_f}{2}.$$

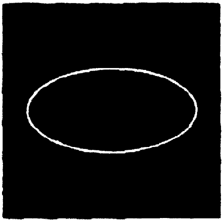

Fig. 17. Axes as 2 : 1.

Radii as 3 : 1 ; $e=0\cdot\dot{3}$.

$a=1,\ p=0\cdot5$; $2^2\epsilon^2=2^2-1^2=3$.

$$2^2y^2\div1^2=2.2\,x_v-x_v^2=2^2-x_c^2$$

$$=1^2+2\sqrt{2^2-1^2}.\,x_f-x_f^2.$$

$$r_v=2.2.1^2\cos\theta_v\div\ 2^2-(2^2-1^2)\cos^2\theta_v\}$$

$$=2.1^2\cos\theta_v\div2\sin^2\psi_v.\quad r_c^2=2^2.1^2\div\{2^2-(2^2-1^2)\cos^2\theta_c\},$$

$$r_c=1\div\sin\psi_c.\quad r_f=1^2\div\{2+\sqrt{2^2-1^2}.\cos\theta_f\}=1^2\div2.2\cos^2\frac{\psi_f}{2}.$$

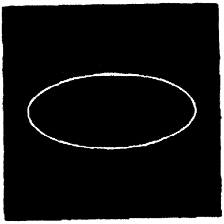

Fig. 18. Axes as 9 : 4.

Radii as 13 : 5; $e=0\cdot3846$.

$a=1,\ p=0\cdot\dot{4}$; $9^2\epsilon^2=9^2-4^2=65$.

$$9^2y^2\div4^2=2.9\,x_v-x_v^2=9^2-x_c^2$$

$$=4^2+2\sqrt{9^2-4^2}.\,x_f-x_f^2.$$

$$r_v=2.9.4^2\cos\theta_v\div\{9^2-(9^2-4^2)\cos^2\theta_v\}$$

$$=2.4^2\cos\theta_v\div9\sin^2\psi_v.\quad r_c^2=9^2.4^2\div\{9^2-(9^2-4^2)\cos^2\theta_c\},$$

$$r_c=4\div\sin\psi_c.\quad r_f=4^2\div\{9+\sqrt{9^2-4^2}.\cos\theta_f\}=4^2\div2.9\cos^2\frac{\psi_f}{2}.$$

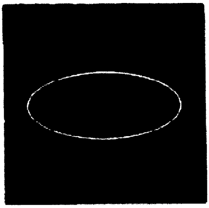

Fig. 19. Axes as 7 : 3.

Radii as 5 : 2 ; $e=0·4$.

$a=1, p=0·\dot{4}2857\dot{1}$; $7^2\epsilon^2=7^2-3^2=40$.

$$7^2y^2\div 3^2=2.7\,x_v-x_v^2=7^2-x_c^2$$
$$=3^2+2\sqrt{7^2-3^2}.x_f-x_f^2.$$

$$r_v=2.7.3^2\cos\theta_v+\{7^2-(7^2-3^2)\cos^2\theta_v\}$$
$$=2.3^2\cos\theta_v+7\sin^2\psi_v. \quad r_c^2=7^2.3^2+\{7^2-(7^2-3^2)\cos^2\theta_c\},$$
$$r_c=3+\sin\psi_c. \quad r_f=3^2+\{7+\sqrt{7^2-3^2}.\cos\theta_f\}=3^2+2.7\cos^2\frac{\psi_f}{2}.$$

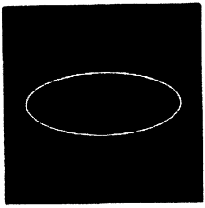

Fig. 20. Axes as 5 : 2.

Radii as 7 : 3 ; $e=0·\dot{4}2857\dot{1}$.

$a=1$, $p=0·4$; $5^2\epsilon^2=5^2-2^2=21$.

$$5^2y^2\div 2^2=2.5x_v-x_v^2=5^2-x_c^2$$
$$=2^2+2\sqrt{5^2-2^2}.x_f-x_f^2.$$

$$r_v=2.5.2^2\cos\theta_v+\{5^2-(5^2-2^2)\cos^2\theta_v\}$$
$$=2.2^2\cos\theta_v+5\sin^2\psi_v. \quad r_c^2=5^2.2^2+\{5^2-(5^2-2^2)\cos^2\theta_c\},$$
$$r_c=2+\sin\psi_c. \quad r_f=2^2+\{5^2+(5^2-2^2)\cos\theta_f\}=2^2+2.5\cos^2\frac{\psi_f}{2}.$$

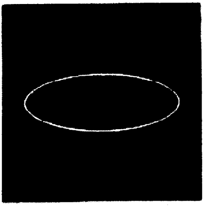

Fig. 21. Axes as 8 : 3.

Radii as 11 : 5 ; $e=0·\dot{4}\dot{5}$.

$a=1$, $p=0·375$; $8^2\epsilon^2=8^2-3^2=55$.

$$8^2y^2+3^2=2.8x_v-x_v^2=8^2-x_c^2$$
$$=3^2+2\sqrt{8^2-3^2}.x_f \quad x_f^2.$$

$$r_v=2.8.3^2\cos\theta\div_v\{8^2-(8^2-3^2)\cos^2\theta_v\}$$
$$=2.3^2\cos\theta_v+8\sin^2\psi_v. \quad r_c^2=8^2.3^2\div\{8^2-(8^2-3^2)\cos^2\theta_c\},$$
$$r_c=3\div\sin\psi_c. \quad r_f=3^2\div\{8+\sqrt{8^2-3^2}.\cos\theta_f\}=3^2\div 2.8\cos^2\frac{\psi_f}{2}.$$

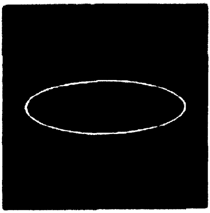

Fig. 22. Axes as 3 : 1.

Radii as 2 : 1; $e=0{\cdot}5$.

$a=1,\ p=0{\cdot}\dot{3}$; $3^2\epsilon^2=3^2-1^2=8$.

$$3^2\,y^2\div1^2=2.3\,x_v-x_v^2=3^2-x_c^2$$

$$=1^2+2\sqrt{3^2-1^2}.x_f-x_f^2.$$

$$r_v=2.3.1^2\cos\theta_v\div\{3^2-(3^2-1^2)\cos^2\theta_v\}$$

$$=2.1^2\cos\theta_v\div3\sin^2\psi_v.\quad r_c^2=3^2.1^2\div\{3^2-(3^2-1^2)\cos^2\theta_c\},$$

$$r_c=1\div\sin\psi_c.\quad r_f=1^2\div\{3+\sqrt{3^2-1^2}\cos\theta_f\}=1^2\div2.3\cos^2\frac{\psi_f}{2}.$$

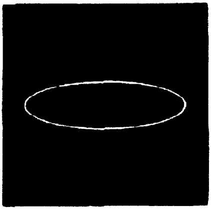

Fig. 23. Axes as 10 : 3.

Radii as 13 : 7; $e=0{\cdot}53846$.

$a=1,\ p=0{\cdot}3$; $10^2\epsilon^2=10^2-3^2=91$.

$$10^2\,y^2\div3^2=2.10\,x_v-x_v^2=10^2-x_c^2$$

$$=3^2+2\sqrt{10^2-3^2}.x_f-x_f^2.$$

$$r_v=2.10.3^2\cos\theta_v\div\{10^2-(10^2-3^2)\cos^2\theta_v\}$$

$$=2.3^2\cos\theta_v\div10\sin^2\psi_v.\quad r_c^2=10^2.3^2\div\{10^2-(10^2-3^2)\cos^2\theta_c\},$$

$$r_c=3\div\sin\psi_c.\quad r_f=3^2\div\{10+\sqrt{10^2-3^2}.\cos\theta_f\}=3^2\div2.10\cos^2\frac{\psi_f}{2}.$$

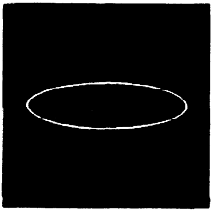

Fig. 24. Axes as 7 : 2.

Radii as 9 : 5; $e=0{\cdot}\dot{5}$.

$a=1,\ p=0{\cdot}\dot{2}8571\dot{4}$; $7^2\epsilon^2=7^2-2^2=45$.

$$7^2y^2\div2^2=2.7x_v-x_v^2=7^2-x_c^2$$

$$=2^2+2\sqrt{7^2-2^2}.x_f-x_f^2.$$

$$r_v=2.7.2^2\cos\theta_v\div\{7^2-(7^2-2^2)\cos^2\theta_v\}$$

$$=2.2^2\cos\theta_v\div7^2\sin^2\psi_v.\quad r_c^2=7^2.2^2\div\{7^2-(7^2-2^2)\cos^2\theta_c\},$$

$$r_c=2\div\sin\psi_c.\quad r_f=2^2\div\{7+\sqrt{7^2-2^2}.\cos\theta_f\}=2^2\div2.10\cos^2\frac{\psi_f}{2}.$$

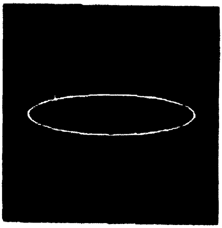

Fig. 25. Axes as 4 : 1.

Radii as $5 : 3$; $e = 0{\cdot}6$.

$a = 1, p = 0{\cdot}25$; $4^2\varepsilon^2 = 4^2 - 1^2 = 15$.

$$4^2 y^2 \div 1^2 = 2.4\,x_v - x_v^2 = 4^2 - x_c^2$$
$$= 1^2 + 2\sqrt{4^2 - 1^2}.\,x_f - x_f^2.$$

$$r_v = 2.4.1^2 \cos\theta_v \div \{4^2 - (4^2 - 1^2)\cos^2\theta_v\}$$
$$= 2.1^2 \cos\theta_v + 4\sin^2\psi_v. \quad r_c^2 = 4^2.1^2 + \{4^2 - (4^2 - 1^2)\cos^2\theta_c\}$$
$$r_c = 1 + \sin\psi_c. \quad r_f = 1^2 + \{4 + \sqrt{4^2 - 1^2}.\cos\theta_f\} = 1^2 + 2.4\cos^2\frac{\psi_f}{2}.$$

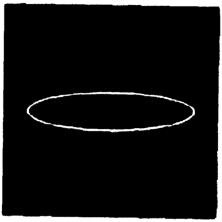

Fig. 26. Axes as 9 : 2.

Radii as $11 : 7$; $e = 0{\cdot}\dot{6}\dot{3}$.

$a = 1,\ p = 0{\cdot}\dot{2}$; $9^2\varepsilon^2 = 9^2 - 2^2 = 77$.

$$9^2 y^2 \div 2^2 = 2.9x_v - x_v^2 = 9^2 - x_c^2$$
$$= 2^2 + 2\sqrt{9^2 - 2^2}.x_f - x_f^2.$$

$$r_v = 2.9.2^2 \cos\theta_v + \{9^2 - (9^2 - 2^2)\cos^2\theta_v\}$$
$$= 2.2^2 \cos\theta_v + 9\sin^2\psi_v. \quad r_c^2 = 9^2.2^2 + \{9^2 - (9^2 - 2^2)\cos^2\theta_c\},$$
$$r_c = 2 + \sin\psi_c. \quad r_f = 2^2 + \{9 + \sqrt{9^2 - 2^2}.\cos\theta_f\} = 2^2 + 2.9\cos^2\frac{\psi_f}{2}.$$

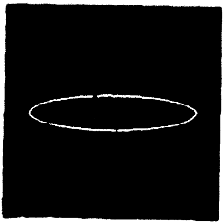

Fig. 27. Axes as 5 : 1.

Radii as $3 : 2$; $e = 0{\cdot}\dot{6}$.

$a = 1, p = 0{\cdot}2$; $5^2\varepsilon^2 = 5^2 - 1^2 = 24$.

$$5^2 y^2 + 1^2 = 2.5x_v - x_v^2 = 5^2 - x_c^2$$
$$= 1^2 + 2\sqrt{5^2 - 1^2}.x_f - x_f^2.$$

$$r_v = 2.5.1^2 \cos\theta_v \div \{5^2 - (5^2 - 1^2)\cos^2\theta_v\}$$
$$= 2.1^2 \cos\theta_v + 5\sin^2\psi_v. \quad r_c^2 = 5^2.1^2 + \{5^2 - (5^2 - 1^2)\cos^2\theta_c\},$$
$$r_c = 1 \div \sin\psi_c. \quad r_f = 1^2 \div \{5 + \sqrt{5^2 - 1^2}.\cos\theta_f\} = 1^2 \div 2.5\cos^2\frac{\psi_f}{2}.$$

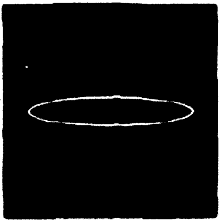

Fig. 28. Axes as 6 : 1.

Radii as 7 : 5 ; $e=0\cdot\dot{7}1428\dot{5}$.

$a=1, p=0\cdot1\dot{6}$; $6^2\epsilon^2=6^2-1^2=35$

$$6^2y^2\div1^2=2.6\,x_v-x_v^2=6^2-x_c^2$$
$$=1^2+2\sqrt{6^2-1^2}.\,x_f-x_f^2.$$

$$r_v=2.6.1^2\cos\theta_v\div\{6^2-(6^2-1^2)\cos^2\theta_v\}$$
$$=2.1^2\cos\theta_v\div6\sin^2\psi_v.\quad r_c^2=6^2.1^2\div\{6^2-(6^2-1^2)\cos^2\theta_c\},$$
$$r_c=1\div\sin\psi_c.\quad r_f=1^2\div\{6+\sqrt{6^2-1^2}.\cos\theta_f\}=1^2\div2.6\cos^2\frac{\psi_f}{2}.$$

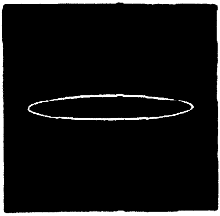

Fig. 29. Axes as 7 : 1.

Radii as 4 : 3 ; $e=0\cdot75$.

$a=1, p=0\cdot\dot{1}4285\dot{7}$; $7^2\epsilon^2=7^2-1^2=48.$

$$7^2y^2\div1^2=2.7\,x_v-x_v^2=7^2-x_c^2$$
$$=1^2+2\sqrt{7^2-1^2}.x_f-x_f^2.$$

$$r_v=2.7.1^2\cos\theta_v\div\{7^2-(7^2-1^2)\cos^2\theta_v\}$$
$$=2.1^2\cos\theta_v\div7\sin^2\psi_v.\quad r_c^2=7^2.1^2\div\{7^2-(7^2-1^2)\cos^2\theta_c\},$$
$$r_c=1\div\sin\psi_c.\quad r_f=1^2\div\{7+\sqrt{7^2-1^2}.\cos\theta_f\}=1^2\div2.7\cos^2\frac{\psi_f}{2}.$$

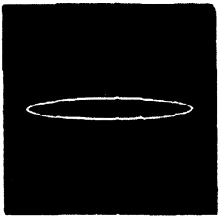

Fig. 30. Axes as 8 : 1.

Radii as 9 : 7; $e=0\cdot\dot{7}$.

$a=1,\ p=0\cdot125$; $8^2\epsilon^2=8^2-1^2=63.$

$$8^2y^2\div1^2=2.8\,x_v-x_v^2=8^2-x_c^2$$
$$=1^2+2\sqrt{8^2-1^2}.x_f-x_f^2.$$

$$r_v=2.8.1^2\cos\theta_v\div\{8^2-(8^2-1^2)\cos^2\theta_v\}$$
$$=2.1^2\cos\theta_v\div8\sin^2\psi_v.\quad r_c^2=8^2.1^2\div\{8^2-(8^2-1^2)\cos^2\theta_c\},$$
$$r_c=1\div\sin\psi_c.\quad r_f=1^2\div\{8+\sqrt{8^2-1^2}.\cos\theta_f\}=1^2\div2.8\cos^2\frac{\psi_f}{2}.$$

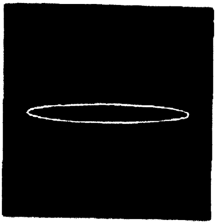

Fig. 31. Axes as 9 : 1.

Radii as 5 : 4; $e=0{\cdot}8$.

$a=1,\ p=0{\cdot}\dot{1}$; $9^2\epsilon^2=9^2-1^2=80$

$$9^2y^2\div1^2=2.9\,x_v-x_v^2=9^2-x_c^2$$
$$=1^2+2\sqrt{9^2-1^2}.x_f-x_f^2.$$

$$r_v=2.9.1^2\cos\theta_v\div\{9^2-(9^2-1^2)\cos^2\theta_v\}$$
$$=2.1^2\cos\theta_v\div9\sin^2\psi_v.\quad r_c^2=9^2.1^2\div\{9^2-(9^2-1^2)\cos^2\theta_c\},$$
$$r_c=1\div\sin\psi_c.\quad r_f=1^2\div\{9+\sqrt{9^2-1^2}.\cos\theta_f\}=1^2\div2.9\cos^2\frac{\psi_f}{2}.$$

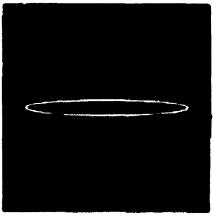

Fig. 32. Axes as 10 : 1.

Radii as 11 : 9; $e=0{\cdot}\dot{8}\dot{1}$.

$a=1,\ p=0{\cdot}1$; $10^2\epsilon^2=10^2-1^2=99$.

$$10^2y^2\div1^2=2.10\,x_v-x_v^2=10^2-x_c^2$$
$$=1^2+2\sqrt{10^2-1^2}.x_f-x_f^2.$$

$$r_v=2.10.1^2\cos\theta_v\div\{10^2-(10^2-1^2)\cos^2\theta_v\}$$
$$=2.1^2\cos\theta_v\div10\sin^2\psi_v.\quad r_c^2=10^2.1^2\div\{10^2-(10^2-1^2)\cos^2\theta_c\},$$
$$r_c=1\div\sin\psi_c.\quad r_f=1^2\div\{10+\sqrt{10^2-1^2}.\cos\theta_f\}=1^2\div2.10\cos^2\frac{\psi_f}{2}.$$

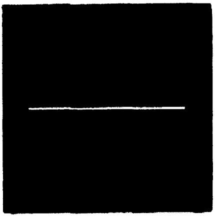

Fig. 33. Axes as 1 : 0.

Radii as 1 : 1; $e=1$.

$a=1,\ p=0$; $1^2\epsilon^2=1^2-0^2=1$ or $\epsilon=1$.

The straight line, the other extreme of the Ellipse, see fig. 1.

Foci at the vertices. $y=0$.

$r_v=r_f=x_v=x_f.\qquad r_c=x_c.$

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